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Surfaces in the hypercubic lattice

E J Janse van Rensburg†

Department of Chemistry, University of Toronto, Toronto, Ontario, M5S 1A1, Canada

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Abstract. Let $s_n(h,g)$ be the number of surfaces, with h boundary components and with genus g, embedded in the hypercubic lattice. Then I prove that for all $h \geqslant 0$ and $g \geqslant 0$ there exists a constant $c_6 > 0$ in two dimensions such that $O(e^{-c_6(\log n)^2})\beta_{1,0}^n \leqslant s_n(h+1,0) \leqslant O(n^h)\beta_{1,0}^n$, and there exist constants $k_7 > 0$ and $k_3 > 0$ such that in three and more dimensions $O(e^{-k_7(\log n)^3})\beta_{1,0}^n \leqslant s_n(h+1,g) \leqslant O(e^{k_3(\log n)^2})\beta_{1,0}^n$, where only k_3 is dependent on h and g, and where $\beta_{1,0}$ is the growth constant of $s_n(1,0)$. In addition, if it is assumed that $s_n(h,g) \sim C(h,g)n^{-\phi_{h,g}}\beta_{1,0}^n$, then I prove that $\phi_{h+1,0} = \phi_{1,0} - h$ in two dimensions, and that $\phi_{h+1,g} \geqslant \phi_{1,0} - 2h - 4g$ in three and more dimensions.

1. Introduction

Surfaces are interesting objects which appear in many different areas of physics, chemistry and mathematics. In quantum field theory they appear in the random surface representation of lattice gauge theories (De Wit and 't Hooft 1977, Fröhlich 1980, Eguchi and Kawai 1982, Kazakov 1983). Interfaces and domain walls are often modelled by surfaces (Binder 1979, Privman and Švrakić 1988), as are membrane-like polymer networks (Maritan and Stella 1984, Kardar and Nelson 1988). More recently, the interest in the properties of vesicles has sparked a renewed interest in surfaces (Fisher *et al* 1991, Banavar *et al* 1991, Mutz and Bensimon 1991). Vesicles are naturally modelled as closed surfaces with a fixed volume *and* a fixed surface area.

We start with the definition of a surface before we consider some previous results and set out the aims of this paper. Let $\mathscr{Z}^d$ be the d-dimensional hypercubic lattice. Let the set of d independent orthogonal unit vectors with endpoints in $\mathscr{Z}^d$ be $\{e_i\}_{i=1}^d$ ($X_j(e_i) = \delta_{ij}$, if $X_j(e_i)$ is the jth cartesian component of the vector e_i). Any *vertex* $v \in \mathscr{Z}^d$ can be represented by a d-tuple $(X_1(v), X_2(v), \ldots, X_d(v))$. An *edge* with endpoints in $\mathscr{Z}^d$ can be represented by the double (v, e_i), if it has endpoints v and $v + e_i$. Similarly, a *plaquette* is a unit square with vertices in $\mathscr{Z}^d$ which one can represent by the triple (v, e_i, e_j) if it has vertices v, $v + e_i$, $v + e_j$ and $v + e_i + e_j$. Two plaquettes are *joined* if they share an edge, and two plaquettes are *connected* if they are elements in a sequence of plaquettes such that neighbouring pairs are joined. A vertex in a set of plaquettes which is pairwise connected is *common* if the plaquettes incident on it are themselves a set of pairwise connected plaquettes. A *surface* is a set of plaquettes such that (i) every edge in the set is incident on either one or two plaquettes, (ii) is pairwise connected, and (iii) has only common vertices. The edges which are incident on only one plaquette each form the *boundary* of the surface. If the surface has no

† Email: bvanrens@alchemy.chem.utoronto.ca

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boundary, then it is *closed.* The boundary is not necessarily connected, but may consist of several components which are called *boundary components.*

Let $\mathscr{S}_n(h)$ be the set of all orientable surfaces in d dimensions with h boundary components consisting of n plaquettes, and let $\mathscr{S}_n = \bigcup_{h\geq 1} \mathscr{S}_n(h)$. Let $s_n(h)$ be the cardinality of $\mathscr{S}_n(h)$ and let $s_n = \Sigma_{h\geq 1} s_n(h)$. In this paper I consider the set $\mathscr{S}_n(h, g)$ of orientable surfaces in d dimensions with h boundary components and genus g. Then $\mathscr{S}_n(h) = \bigcup_{g\geq 0} \mathscr{S}_n(h, g)$. The cardinality of $\mathscr{S}_n(h, g)$ is represented by $s_n(h, g)$. It was shown that $\lim_{n\to\infty} s_n^{1/n} = \beta$, $\lim_{n\to\infty} s_n(h)^{1/n} = \beta_h$ and that $\lim_{n\to\infty} s_n(h, g)^{1/n} = \beta_{h,g}$ (Janse van Rensburg and Whittington 1989, 1990, hereafter referred to as JRW1 and JRW2). It was also shown that β_h is independent of h and that $\beta_{h,g}$ is independent of h and of g.

The analogous number to $s_n(h, g)$ for the self-avoiding walk is c_n (the number of self-avoiding walks in the hypercubic lattice, rooted at the origin), which has been the subject of numerous investigations. Hammersley and Morton (1954) proved the existence of a connective constant κ, such that $\lim_{n\to\infty} c_n^{1/n} = e^{\kappa}$ (e^{κ} is also called the *growth constant*). c_n is a submultiplicative function of n, i.e. $c_n c_m \geq c_{n+m}$. Consequently, by the theory of subadditive functions, one notes that $c_n \geq \mu^n = e^{\kappa n}$ (Hille 1948). A remarkable effort by Hammersley and Welsh (1962) established that $\mu^n \leq c_n \leq \mu^n O(e^{\gamma\sqrt{n}})$. These bounds were improved by Kesten (1964). It is widely believed that c_n deviates from exponential growth by at most a power of n. In more than four dimensions this has now been proven to be indeed the case (Hara and Slade 1991). In four and less dimensions this fact has not been established.

In analogy to c_n I shall calculate bounds on $s_n(h, g)$. These new bounds are the main new results in this paper. The bounds are consequences of two new constructions, *unfolding*, and a new *submultiplicative inequality* involving $s_n(1, 0)$ (propositions 2.3 and 3.1). In particular, I found a positive constant c_6 such that in two dimensions (theorem 3.4(ii) and (iii))

$$O(e^{-c_6(\log n)^2})\beta_{1,0}^n \leq s_n(h+1, 0) \leq O(n^h)\beta_{1,0}^n \qquad \forall h \geq 0 \tag{1.1}$$

and in three and more dimensions I prove that there exists positive constants k_3 (dependent only on h and g) and k_7 such that (theorems 3.9(iii) and 3.10)

$$O(e^{-k_7(\log n)^3})\beta_{1,0}^n \leq s_n(h+1, g) \leq O(e^{k_3(h,g)(\log n)^2})\beta_{1,0}^n \qquad \forall h \geq 0 \quad \text{and} \quad g \geq 0. \tag{1.2}$$

It is usually assumed (in analogy with walks and animals, and on the basis of numerical results) that the corrections to the exponential behaviour of $s_n(h, g)$ (or $s_n(h)$) is dominated by a power law. That is, $s_n(h) \sim C_h n^{-\phi_h} \beta_1^n$. The exponents ϕ_h were found to obey the following relations in JRW1 and JRW2: in two dimensions

$$\phi_h \geq \phi_{h+1} \quad \text{and} \quad \phi_1 - h \geq \phi_{h+1} \geq \phi_1 - \tfrac{3}{2}h \qquad \forall h \geq 1 \tag{1.3}$$

and in three and more dimensions

$$\phi_h \geq \phi_{h+1} \geq \phi_1 - 2h \qquad \forall h \geq 1. \tag{1.4}$$

In this paper I assume that $s_n(h, g) \sim C_{h,g} n^{-\phi_{h,g}} \beta_{1,0}^n$ (where $\phi_{h,0} = \phi_h$ in the two-dimensional case). I improve (1.3) to the *equality* (theorem 3.5(i))

$$\phi_{h+1,0} = \phi_{1,0} - h \qquad \forall h \geq 0 \tag{1.5}$$

and in three and more dimensions I prove a relation between exponents (which was given without proof in JRW2):

$$\phi_{h-\mu,g-\nu} \geq \phi_{h+1,g} \geq \phi_{1,0} - 2h - 4g \qquad \forall h \geq 0 \quad \text{and} \quad g \geq 0 \tag{1.6}$$

where μ and ν are any integers greater or equal to zero (theorem 3.14).

Often, surfaces are studied in the mean field approximation (Fröhlich 1985) which provides 'exact' values of the critical exponents above the critical dimension, which is eight. In lower dimensions, mean field theory is inadequate, and one has to resort to other methods to derive some of the properties of surfaces. There is strong evidence (numerical and rigorous) that surfaces collapse to branched polymers in the scaling limit (Bovier *et al* 1984, Fröhlich 1985, O'Connell *et al* 1991). The values one would expect for the exponent $\phi_{1,0}$ are therefore the tree exponents. In particular, one can find 'exact' values for $\phi_{1,0}$ from the dimensional reduction of Parisi and Sourlas (1981) in two dimensions ($\phi_{1,0}=1$) and in three dimensions ($\phi_{1,0}=\frac{3}{2}$). In higher dimensions one can use Flory's argument (Isaacson and Lubensky 1980) to guess that $\phi_{1,0}$ increases with increasing dimension to its mean field value ($\frac{5}{2}$) in eight dimensions. In the previous works (JRW1 and JRW2), and in this paper, a combinatorial approach to surfaces is developed. These methods are well established for walks (Hammersley 1957, 1961a, 1961b, 1962, Hammersley and Welsh 1962) and can naturally be applied to surfaces.

This paper is organized in the following way: in section 2 I consider the unfolding of surfaces, a construction which will prove very useful in other applications later in this work. I also review the basic definitions, and recall some useful results from JRW1 and JRW2. In section 3 I consider surgery on surfaces. In particular, I prove a 'submultiplicative' property for surfaces (proposition 3.1). In section 3.1 I study surfaces in two dimensions. I recall some useful results from JRW1 and tighten a bound first proved in JRW1. Together with unfolding and the submultiplicative property, these propositions reveal interesting bounds on $s_n(h, 0)$ in two dimensions. In section 3.2 I switch my attention to surfaces in three and more dimensions. I generalize some inequalities in JRW1 and JRW2 and study the properties of $s_n(h, g)$ in light of unfolding and the submultiplicative property. Section 4 is given to some conclusions and speculations about future directions.

2. Unfolded surfaces

The (unique) *top vertex* and the (unique) *bottom vertex* of a surface $\sigma\in\mathscr{S}_n$ are found by a lexicographic ordering of all the vertices in the surface. To find the *top edge* and the *bottom edge* of σ one must first consider the following definition: an edge (t, e_i) is *perpendicular* to a second edge (t', e_j) if $e_i\cdot e_j=0$. Since the top and bottom vertices of σ are incident on at least one plaquette each, there must be at least two edges incident on the top vertex (and on the bottom vertex). Without loss of generality, consider only the top vertex, t. Let the set of edges incident on t be $\{(t, -e_{i_j})\}_{j=1}^{m}$, where $m\geqslant 2$. (The minus sign appears from the definition of the top vertex.) Let $k=\min_j\{i_j \mid k\neq 1\}$. Since $m\geqslant 2$ there is always such a k. Then the top edge of σ is $(t, -e_k)$. It is perpendicular to $(t, -e_1)$. The bottom edge is found in a similar way. Let the boundary of σ be $\partial\sigma$. Let $H_{ij}(z)=\{x\in\mathscr{R}^d \mid X_i(x-z)=X_j(x-z)\}$ be a hyperplane containing the point z.

Definition 2.1. A surface σ is said to be *top unfolded* if its top edge is in $\partial\sigma$ or if there exists a hyperplane $H_{ij}(t)$ (with $i,j>1$) containing the top vertex t such that a reflection of σ through this hyperplane, σ', has its top edge in $\partial\sigma'$. If the bottom edge is in $\partial\sigma$, or if there exists a hyperplane $H_{ij}(b)$ (with $i,j>1$) containing the bottom vertex b such that a reflection of σ through this hyperplane, σ'', has its bottom edge in $\partial\sigma''$, then it is *bottom unfolded*. A surface which is both top and bottom unfolded is said to be *doubly unfolded*.

Let the set of all top unfolded surfaces with n plaquettes be $\mathcal{S}^t_n$, the set of all bottom unfolded surfaces with n plaquettes be $\mathcal{S}^b_n$, and the set of doubly unfolded surfaces with n plaquettes be $\mathcal{S}^*_n (=\mathcal{S}^t_n \cap \mathcal{S}^b_n)$. Let the cardinalities of these sets be s^t_n, s^b_n and s^*_n respectively. I shall study surfaces with a fixed number of boundary components, h, and a fixed genus g. Let the set of all surfaces with h boundary components, genus g, and n plaquettes be $\mathcal{S}_n(h, g)$. I extend the notation to indicate the set of all top unfolded surfaces with h boundary components, genus g, and n plaquettes by $\mathcal{S}^t_n(h, g)$. This set has cardinality $s^t_n(h, g)$. Similar notation applies to bottom unfolded surfaces and to doubly unfolded surfaces. By symmetry, $s^t_n(h, g) = s^b_n(h, g)$. In any equation one can replace s^t_n by s^b_n, or $s^t_n(h, g)$ by $s^b_n(h, g)$. Define $s^u_n = s^t_n = s^b_n$ and $s^u_n(h, g) = s^t_n(h, g) = s^b_n(h, g)$. In two dimensions, every surface is doubly unfolded, and there are no surfaces with genus bigger than zero (i.e. $s_n(h, g) = 0$ if $d = 2$ and $g > 0$). The following inequalities are easily proven by appending plaquettes onto the top edge (or bottom edge) of an unfolded surface:

Proposition 2.2. Let $d \geq 2$. Then

(i) $s^u_n(h, g) \leq s^u_{n+1}(h, g)$

(ii) $s^u_n(h, g) \leq s^u_{n+C}(h+1, g)$

(iii) $s^u_n(h, g) \leq s^u_{n+K}(h, g+1)$

where C and K are constants independent of n, and where one can also replace u by $*$.

Let $\lfloor x \rfloor$ be the largest integer smaller or equal to x, and let $\lceil x \rceil$ be the smallest integer larger or equal to x. In more than two dimensions, one finds the following relations between surfaces and unfolded surfaces:

Proposition 2.3. Let $d \geq 3$. Then there exist a map from $\mathcal{S}_n$ into $\mathcal{S}^t_n$ or $\mathcal{S}^b_n$ which is at most $2n^{\lceil \log n/\log 4 \rceil}$ to 1, and there exist a map from $\mathcal{S}_n$ to $\mathcal{S}_n^*$ which is at most $4n^{2\lceil \log n/\log 4 \rceil}$ to 1. In addition, if $\sigma \in \mathcal{S}_n$ is mapped onto σ', and σ has h boundary components and genus g, then σ' has *at most h* boundary components and has genus g. Consequently,

(i) $s_n \leq 2n^{\lceil \log n/\log 4 \rceil} s^u_n$,

(ii) $s_n \leq 4n^{2\lceil \log n/\log 4 \rceil} s^*_n$,

(iii) $s_n(h, g) \leq 2n^{\lceil \log n/\log 4 \rceil} h s^u_{n+(h-1)C}(h, g)$, and

(iv) $s_n(h, g) \leq 4n^{2\lceil \log n/\log 4 \rceil} h^2 s^*_{n+2(h-1)C}(h, g)$,

where C is a constant independent of n.

Proof. Let $\sigma \in \mathcal{S}_n$. Suppose that σ is not top unfolded. Let the top edge of σ be $(t, -e_k)$, then $(t, -e_k) \notin \partial\sigma$. Starting at the top edge, mark the shortest sequence of plaquettes, successively joined, which connects the top edge to an edge in $\partial\sigma$. Let this set of plaquettes be ρ. ρ is a subsurface, and define $\sigma' = \sigma - \rho$. Then σ' has its top edge in $\partial\sigma'$. Reflect the surface ρ through the midpoint of the top edge, $(t - e_k)/2$, and consider the surface $\sigma' \cup \rho'$, where ρ' is the reflected image of ρ. The maximum number of plaquettes in ρ is $\lceil n/4 \rceil$. To see this, delete first one of the plaquettes incident on the top edge of σ. Let p be the number of plaquettes in ρ after this plaquette is deleted. Observe that the hypercubic lattice has girth four, and that alternative sets of plaquettes adjacent to ρ can instead be identified. A fourth set of plaquettes adjacent to one of these sets can also be identified. By definition, each one of these sets has at least p plaquettes. Therefore, $4p \leq n-1$. Hence, ρ has at most $(n-1)/4+1$ plaquettes. The number of plaquettes in ρ must be an integer, therefore, the maximum number is bounded by $\lfloor (n-1)/4+1 \rfloor = \lceil n/4 \rceil$ (see figure 1).

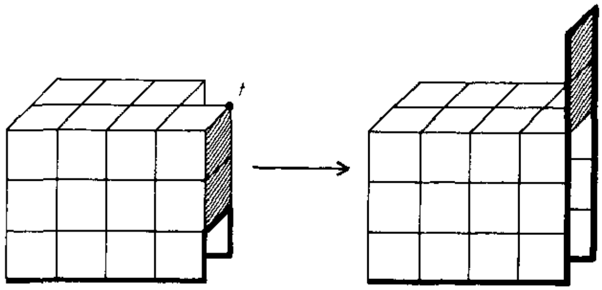

Figure 1. Unfolding of a surface. ρ, the shaded section of the surface, is reflected through the top edge.

The surface $\sigma' \cup \rho'$ has a new top edge, which is in ρ'. The bottom edge of ρ' is $(t, -e_k)$, by the construction. Since the bottom edge of ρ' is the only connection between σ' and ρ', ρ' is bottom unfolded. Identify a new top edge on ρ', and repeat the construction in the paragraph above. Look for the smallest subsurface ρ'' which connects the top edge to the boundary. If ρ'' contains the bottom edge of ρ', then it must have at least $n - \lceil n/4 \rceil$ plaquettes, this is a contradiction (since it contains more than $n/2$ plaquettes). Thus, $\rho'' \subset \rho'$, and ρ'' cannot contain the plaquette incident on the bottom edge of ρ'. The maximum number of plaquettes in ρ'' is $\lceil \lfloor n/4 \rfloor \rceil$. To see this, note that ρ'' does not contain the plaquette incident on the bottom edge of ρ'. Delete one plaquette incident on the top edge of ρ', and suppose that ρ'' has $p+1$ plaquettes. Then by the arguments in the previous paragraphs, $4p \leq \lceil n/4 \rceil - 2$. But the maximum number of plaquettes in ρ'' is $p+1$, an integer. Thus, $p+1 \leq \lfloor \lceil n/4 \rceil/4 + 1/2 \rfloor \leq \lceil \lfloor n/4 \rfloor /4 \rceil$.

Suppose that after m applicatons of this construction the surface is top unfolded (it has its top edge on its boundary). At the mth application, the number of plaquettes involved in the reflection is at most (as is easily seen by performing an induction) $\lceil \lfloor \lfloor \ldots \lfloor n/4 \rfloor /4 \ldots \rfloor /4 \rfloor /4 \rceil$ (m divisions by 4). If the number of plaquettes at the $(m+1)$th application is less than 1, then the surface must be top unfolded at the mth application. Hence, $\lceil \lfloor \ldots \lfloor n/4 \rfloor /4 \ldots /4 \rfloor /4 \rceil = 1$ (where there are m divisions by 4). That is, m is at most the biggest integer such that $n \geq 4^m$, or $m \leq \log n / \log 4$. Let the unfolded surface be σ^u. The unfolded surface σ^u is obtained by applying the construction above at most $\lceil \log n / \log 4 \rceil$ times to the surface. At the ith unfolding, the top edge can be found in at most $\lceil \lfloor \ldots \lfloor n/4 \rfloor /4 \ldots \rfloor /4 \rceil \leq \lceil n/4^i \rceil$ locations (i divisions by 4) since there are at most one top edge for every plaquette. Thus, the unfolding maps at most $z_n = \prod_{i=1}^{\lceil \log n/\log 4 \rceil} \lceil n/4^i \rceil$ surfaces to σ^u. But $z_n \leq \prod_{i=1}^{\lceil \log n/\log 4 \rceil} [n(1/n + 4^{-i})] \leq n^{\lceil \log n/\log 4 \rceil} \prod_{i=0}^{\infty} (1+4^{-i}) \leq 2n^{\lceil \log n/\log 4 \rceil}$ (Gradshteyn and Ryzhik 1965). Therefore, $s_n \leq 2n^{\lceil \log n/\log 4 \rceil} s_n^u$, which is the inequality claimed in (i). The second inequality is found by unfolding to the top and to the bottom.

From a topological point of view an application of the construction consists of making two cuts starting at the endpoints of the top edge and ending on either one or two boundary components. The segment between the cuts is then folded out. If both cuts terminate on the same boundary component, then the surface has the same topology after application of the construction. If they end on two different components, then the components coalesce when one folds the segment out; this reduces the number of boundary components by 1. The genus of the surface cannot be changed; to do that, one must cut the surface along a 1-cycle which is not a boundary (in the homology sense). Therefore, if $\sigma \in \mathscr{S}_n(h, g)$, then σ^u is a surface in the set $\bigcup_{i=1}^{h} \mathscr{S}_n(i, g)$. Thus $s_n(h, g) \leqslant 2n^{\lceil \log n/\log 4 \rceil} \sum_{i=1}^{h} s_n^u(i, g)$. If one applies the results from proposition 2.2, then the inequality claimed in (iii) is found. The last inequality is found by unfolding the surfaces to the top and to the bottom.

A top unfolded surface can be concatenated with a bottom unfolded surface if their top and bottom edges (respectively) have the same orientation. This gives the following set of useful inequalities:

Proposition 2.4. Let $d \geqslant 2$. By concatenating surfaces one deduces that

(i) $s_n^u s_m^u \leqslant (d-1) s_{n+m}$

(ii) $\sum_{h'=1}^{h-1} \sum_{g'=0}^{g} s_n^u(h', g') s_m^u(h-h', g-g') \leqslant (d-1) s_{n+m}(h-1, g)$

(iii) $s_n^* s_m^* \leqslant (d-1) s_{n+m}^*$ and

(iv) $\sum_{h'=1}^{h-1} \sum_{g'=0}^{g} s_n^*(h', g') s_m^*(h-h', g-g') \leqslant (d-1) s_{n+m}^*(h-1, g)$.

Proof. If $d=2$, then note that $s_n^t(h, 0) = s_n^b(h, 0) = s_n^*(h, 0) = s_n(h, 0)$. Concatenate surfaces in pairs to find each of the resulting inequalities. If $d \geqslant 3$ then by symmetry $s_n^t(h, g) = s_n^b(h, g)$. Let $\sigma_1 \in \mathscr{S}_n^t(h_1, g_1)$ and $\sigma_2 \in \mathscr{S}_m^b(h_2, g_2)$. If the top edge of σ_1 has the same orientation as the bottom edge of σ_2, then one can concatenate them to find a new surface $\sigma_1 \oplus \sigma_2$ with $n+m$ plaquettes, $h_1 + h_2 - 1$ boundary components, and genus $g_1 + g_2$. Suppose therefore that the top edge of σ_1 is $(t, -e_k)$, and that the bottom edge of σ_2 is (b, e_l). Consider the hyperplane $H_{kl}(b) = \{x \in \mathscr{R}^d \mid X_k(x-b) = X_l(x-b)\}$, which contains b. If one reflects σ_2 through $H_{kl}(b)$, then the new surface will still have bottom vertex b, but the bottom edge is now (b, e_k). This reflection maps all the surfaces σ_2 into a set of surfaces which has bottom edge (b, e_k). The reflection is at most $(d-1)$ to 1.

Remarks. If one sums proposition 2.4(ii) over g, then $\sum_{h'=1}^{h-1} s_n^u(h', *) s_m^u(h-h', *) \leqslant (d-1) s_{n+m}(h-1, *)$, where $s_n(h, *) = \sum_{g=0}^{\infty} s_n(h, g)$ $(= s_n(h))$. Alternatively, if one sums proposition 2.4(ii) over h, then $\sum_{g'=0}^{g} s_n^u(*, g') s_m^u(*, g-g') \leqslant (d-1) s_{n+m}(*, g)$, where $s_n(*, g) = \sum_{h=1}^{\infty} s_n(h, g)$. Similar inequalities can be found by summing proposition 2.4(iv) over h or g. If one sums proposition 2.4(ii) over both h and g, then one obtains proposition 2.4(i), since $s_n(*, *) = s_n$. Similarly, one can get proposition 2.4(iii) from porposition 2.4(iv) by summing over both h and g. I conclude this section by noting the following useful results:

Proposition 2.5. (JRW1 and JRW2) Let $d \geqslant 2$. Then there exist constant, positive integers A, H and G such that

(i) $s_n(h, g) \leqslant s_{n+A}(h, g)$ (and consequently $s_n \leqslant s_{n+A}$),

(ii) $s_n(h, g) \leqslant s_{n+H}(h+1, g)$, and

(iii) $s_n(h, g) \leqslant s_{n+G}(h, g+1)$.

Proposition 2.6. (Durhuus *et al* 1983, JRW1) Let $d \geqslant 2$. Then there exists a finite, positive constant κ such that $s_n \leqslant \kappa^n$.

3. Surgery on surfaces

Surgery on surfaces involves the cutting away of segments of a surface in order to change the number of boundary components, or the genus, or to separate the surface into more than one segment. The objective is to relate $s_n(h, g)$ to $s_n(h-1, g)$ or $s_n(h, g-1)$ in very much the same fashion as in proposition 2.2. In proposition 2.4 it was noted that surfaces naturally satisfy a supermultiplicative equation. It is remarkable that one can prove a submultiplicative equation for discs (orientable surfaces with one boundary component and zero genus):

Proposition 3.1. Suppose that $\sigma \in \mathscr{S}_{n+m}(1,0)$. Then σ can be partitioned into two pieces such that one piece has n plaquettes, and the other has a total of m plaquettes. This construction leads to the following inequalities:

(i) $s_{n+m}(1,0) \leqslant (\alpha_0(n+m))^{3\lceil \log(n+m)/\log(3/2)\rceil} s_n(1,0) s_m(1,0)$ if $d=2$, and

(ii) $s_{n+m}(1,0) \leqslant (\alpha_1(d-1)(n+m)^{3+2\lceil \log(n+m)/\log 4\rceil})^{\lceil \log(n+m)/\log(3/2)\rceil} s_n(1,0) s_m(1,0)$ if $d \geqslant 3$,

where α_0 and α_1 are two fixed positive constants.

Proof. Suppose that there exists a plaquette in σ with the property that if deleted, it will separate σ in at least 2 (and as many as 4) pieces. These plaquettes are called *split-plaquettes.* Suppose that σ has no split-plaquettes. Then one can partition σ into 2 pieces, one with m plaquettes. To see this, suppose that σ has been partitioned into $n-(m-1)$ and $m-1$ plaquettes, and let l be the boundary curve between the two pieces. Let the two pieces be $\rho(n-m+1)$ and $\rho(m-1)$. Select a plaquette p on $\rho(n-m+1)$ with at least one edge in l. One can add p to $\rho(m-1)$ if p is not a split-plaquette in $\rho(n-m+1)$. If p is a split-plaquette in $\rho(n-m+1)$, then it must partition the subsurface into two pieces, one with (say) j plaquettes, and the other with $n-m-j$ plaquettes, where, without loss of generality, $j \leqslant n-m+1-j$. If $j=0$, the p is not a split-plaquette of $\rho(n-m+1)$, so suppose that $j>0$. Select a second plaquette p' in the piece with j plaquettes, with p' incident on l (observe that p' can always be chosen this way, if this is not possible, then p is a split-plaquette of σ). One can add p' to $\rho(m-1)$ unless it is a split-plaquette, otherwise, one observes that it partitions $\rho(n-m+1)$ into two segments, one with (say) k plaquettes, where $k<j$. If $k=0$, then one can immediately add p' to $\rho(m-1)$. If $k>0$, then put $j=k$ and repeat the process, finding a new k. Eventually, $k=0$, and one can add the last selected plaquette to $\rho(m-1)$. By induction, one can then partition a disc into two segments of arbitrary size, as long as there are no split-plaquettes in the disc.

Observe that one can easily adapt this constructive proof to show that if a disc (with n plaquettes) connects two split-plaquettes of a surface, then one can partition the disc into two arbitrary pieces, such that each piece contains exactly one split-plaquette, provided that the union of the disc and the split-plaquettes is a disc without split-plaquettes. To do this, choose one split-plaquette as the first plaquette in the partition. Suppose that one has marked $m-1$ plaquettes, and is looking for the mth plaquette. Choose it on the curve l as before, and add it if it is not a split-plaquette of $\rho(n-m+1)$. If it is, then observe that it splits $\rho(n-m+1)$ into two segments, one of which contains the second split-plaquette. Select the segment without the second split-plaquette, and suppose that it has j plaquettes. The argument is now as before, except that one always selects the piece not containing the second split-plaquette.

I now prove that one can always cut a segment with q plaquettes from a disc σ with n plaquettes, where $\lceil m/3 \rceil \leqslant q \leqslant m$ (and where $1 \leqslant m \leqslant n$ and $q>0$). Let u_i, $i=1,2,\ldots$, be a sequence of integers. Then one defines $\Sigma^j u_i = \Sigma_{i=1}^{j} u_i$. If σ contains no split-plaquettes, then we are finished by the above construction. So suppose that σ has at least one split-plaquette p (see figure 2). Incident on p are four pieces of the disc, which has t_i plaquettes each, $1 \leqslant i \leqslant 4$, and where some of the t_i might be zero. Observe that $\Sigma^4 t_i = n-1$. Without loss of too much generality one can suppose that $m \leqslant \lfloor n/2 \rfloor$ (if $m \geqslant \lceil n/2 \rceil$ consider $n-m$ instead; since m and $n-m$ will then be treated

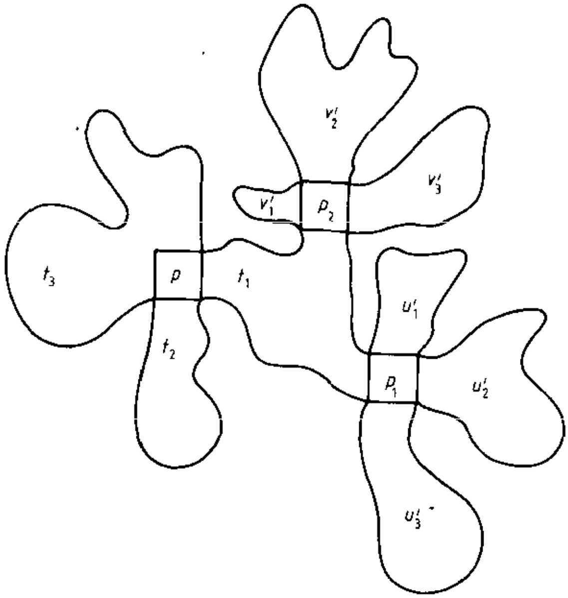

Figure 2. Towards a submultiplicative inequality for discs.

in the same way, this introduces a factor of 2 in our results). Suppose that $t_i < m/2$ for all i. Then $n-1 = \Sigma^4 t_i < 2m \leqslant n$. This is a contradiction, unless $2m = n$ (and $t_i < m/2$). Hence, $\Sigma^4 t_i = 2m-1$. The maximum value of $\Sigma^4 t_i$ (under our assumptions) is attained when $t_i = t_j$, $\forall i$ and j. If m is even, then $t_i = m/2-1$. Hence $\Sigma^4 t_i = 2m-4$. This is a contradiction. If m is odd, then $t_i = m/2 - \frac{1}{2}$. Hence $\Sigma^4 t_i = 2m-2$, another contradiction. Hence, there exists at least one i such that $t_i \geqslant m/2$. That is $q = t_i \geqslant \lceil m/3 \rceil$.

If $q \leqslant m$ then the process is finished (use the construction on partitioning a disc with two split-plaquettes). So suppose that $q > m$, and without loss of generality, suppose that $q = t_{i_1}$, for some i_1. Consider now the subsurface consisting of t_{i_1} plaquettes, which is connected to the rest of the surface at the split-plaquette p. If there are no other split-plaquettes in the subsurface, then one can partition m plaquettes from it using the construction above. Otherwise, suppose that there are other split-plaquettes in the surface, and suppose that p_1 is one of these. I show now that one can either separate a segment of size between $\lceil m/3 \rceil$ and m from σ, or one can find a subsurface connected to the rest of the surface at exactly one split-plaquette and which contains t_{i_2} plaquettes, such that $t_{i_1} > t_{i_2} > \lceil m/3 \rceil$. If this construction is iterated, then one must finally find a subsurface of the desired size. Incident on p_1 are (at most) four segments of σ, one which contains p (the first split-plaquette considered). Let this piece contain u_4' plaquettes, and let the other pieces contain u_i', $1 \leqslant i \leqslant 3$, plaquettes. There are two cases: case 1 which has $\Sigma^3 u_i' \geqslant m$, or case 2 which has $\Sigma^3 u_i' < m$. In case 1 there exists an i such that $u_i' \geqslant \lceil m/3 \rceil$. So let $t_{i_2} = u_i'$, and observe that $t_{i_1} > t_{i_2}$. Under case 2 there are two subcases: either one has $\Sigma^3 u_i' \geqslant \lceil m/3 \rceil - 1$, or $\Sigma^3 u_i' < \lceil m/3 \rceil - 1$. In the first subcase observe that adding the split-plaquette p_1 to the three segments results in a connected subset of σ of size at least $\lceil m/3 \rceil$, and we are done. In the second subcase one must add plaquettes beyond p_1 until the size is more or equal to $\lceil m/3 \rceil$. The construction is similar to separating the surface between p_1 and p. There is one difference: some of the plaquettes added could be split-plaquettes which may increase the size of the subsurface from below $\lceil m/3 \rceil$ to above m. If this happens, let p_2 be that split-plaquette. Obviously, $p_2 \neq p$; (if $p_2 = p$, then $q = t_{i_1} < \lceil m/3 \rceil$, which is a contradiction). Incident on p_2 are at most 4 segments of the surface, one of which contains p. Let the other pieces incident on p_2 be v_i', where $1 \leqslant i \leqslant 3$ (If p_1 is not on the same segment as p, then it is contained in one of these segments.) If p_1 is on the same segment as p (see figure 2), then $t_{i_1} > t_{i_2} = \Sigma^3 v_i' + 1 > m - \lceil m/3 \rceil \geqslant \lceil m/3 \rceil$. If p_1 is in another segment, say (without loss of generality), in v_3', then $\Sigma^2 v_i' + 1 > m - \lceil m/3 \rceil = \lfloor 2m/3 \rfloor$. To satisfy this constraint, at least one of v_1' and v_2' must be greater or equal to $\lceil m/3 \rceil$. Choose t_{i_2} to be this segment. Then $t_{i_1} > t_{i_2} > \lceil m/3 \rceil$.

Once a subsurface with q plaquettes has been partitioned from σ, where $\lceil m/3 \rceil \leqslant q \leqslant m$, one can put $m^{(1)} = m - q$ and consider the original surface minus the partitioned segment. If the construction is repeated, then one can partition a second subsurface with at least $\lceil m^{(1)}/3 \rceil$ (and at most $m^{(1)}$) plaquettes. This process stops when one has partitioned out in total m plaquettes. The maximum number of partitionings occur when one separates out the minimum number of plaquettes at every stage. At the kth application one separates out $\lceil \lfloor 2 \ldots 2 \lfloor 2m/3 \rfloor \ldots /3 \rfloor /3 \rceil$ plaquettes, where there are $(k-1)$ multiplications by 2 and k divisions by 3. Hence, the partitioning stops when $\lceil \lfloor \ldots 2 \lfloor 2m/3 \rfloor /3 \rfloor \ldots /3 \rfloor /3 \rceil = 1$ after $k-1$ applications. Then one can easily check that $m \geqslant 2(3/2)^k$, or $k \leqslant \log m/\log(3/2)$. The maximum number of ways that one can attempt to put anyone of the pieces back in the surface is bounded by $\alpha_0 nm$, where α_0 is a positive constant. In two dimensions, concatenate the segments into a surface with m plaquettes, using proposition 2.4. By the arguments in proposition 2.3 one can partition this surface into k pieces in at most m^k ways. Hence, if one put $m = i$ and $n - m = j$, and absorbs the extra factor of 2 (which arises because m and $n - m$ are treated in the same way) into α_0, then $s_{i+j}(1,0) \leqslant (\alpha_0 i^2 j)^{\log i/\log(3/2)} s_i(1,0) s_j(1,0)$. This is better than the inequality claimed in two dimensions. In three and higher dimensions one cannot concatenate the pieces as claimed. Unfold them instead, using proposition 2.3, and concatenate them using proposition 2.4. The result is $s_{i+j}(1,0) \leqslant (\alpha_1 j i^{2+2\lceil \log i/\log 4 \rceil})^{\log i/\log(3/2)} s_i(1,0) s_j(1,0)$, where α_1 is a positive constant (which has absorbed the extra factor of 2 arising from the similiar treatment given to n and $n-m$). This is better than the inequality claimed in three and more dimensions.

3.1. Surgery in two dimensions

A surface in two dimensions always has genus zero; in this subsection we ignore the genus of the surfaces and concentrate on the number of boundary components. In JRW1, a stronger result than proposition 2.2(ii) was proven in two dimensions. I begin by quoting that result:

Proposition 3.2. (JRW1) Let $d = 2$. Then there exists finite, non-zero constants c_0, c_1 and c_2 such that

$$\binom{\lfloor \frac{n}{c_0} \rfloor}{h} s_n(1,0) \leqslant c_1^h s_{n+hc_2}(h+1,0).$$

This proposition is a pattern theorem for surfaces in two dimensions similar to the pattern theorem for walks and animals (Kesten 1963, Madras 1989). Surgery on surfaces are performed by cutting between boundary components.

Proposition 3.3. Let $d = 2$. Then there exists a finite, positive constant c_3 such that

$$s_n(h+1,0) \leqslant 2n s_{n+c_3}(h,0).$$

Proof. Let $\sigma \in \mathscr{S}_n(h, 0)$, where $h \geq 2$. Suppose that $\partial\sigma$ has one boundary component which is also the boundary of a plaquette. Then one can delete this boundary component by adding the plaquette to the surface. This construction is a map into $\mathscr{S}_{n+1}(h-1, 0)$, and is at most n to 1, since one can find at most n locations on $\mathscr{S}_{n+1}(h-1, 0)$ to construct a new boundary component. If $\partial\sigma$ has no components which are boundaries of plaquettes, then argue as follows: Let $\mathscr{D}$ be the surface in $\mathscr{S}_9(1, 0)$ which has a square as boundary. Let x and y be points on different boundary components of σ, and select x and y to be only the midpoints of edges on the boundary components. Connect x to y by a curve which passes through the midpoints of every plaquette it traverses, and which passes from one plaquette to the next only through an edge (and not through a vertex). The length of this curve, C_{xy}, is defined as the number of midpoints of plaquettes that it visits. Vary x and y over $\partial\sigma$ to find the curve C_{xy} which has the minimum length. Let ρ be the segment of σ which consists of the plaquettes traversed by C_{xy}. ρ is a linear string of plaquettes with every plaquette having at least two edges in $\partial\rho$. To see this, one must show that one does not disconnect σ when ρ is deleted. Let the 'plaquette at z' mean 'the plaquette with midpoints z'. Without loss of generality, suppose that plaquettes at $z-e_1$ and z have been deleted, and that the next plaquette to be deleted is the plaquette at $z+e_1$. If the surface is disconnected when $z+e_1$ is deleted, then $z+2e_1+e_2$ and $z+2e_1-e_2$ must be unoccupied, while $z+2e_1$ is occupied. In that case delete $z+e_2$ instead. If this disconnects the surface, then $z+e_1+2e_2$ and $z-e_1+2e_2$ must be occupied, while $z+2e_2$ is occupied. But this is not possible; if $z-e_1+2e_2$ is unoccupied, then delete $z-e_1+e_2$, which will be a shorter path to the boundary. Thus, $z-e_1+2e_2$ must be occupied. Therefore, one can delete $z+e_2$ and then $z+e_1+e_2$ on the boundary, without disconnecting σ. Therefore, ρ is a linear string of plaquettes. Suppose that ρ has j plaquettes, and define $\sigma' = \sigma - \rho$ with $n-j$ plaquettes, and $h-1$ boundary components. Concatenate $\sigma' \oplus \mathscr{D} \oplus \rho = \tau$, where τ has $h-1$ boundary components and $n+9$ plaquettes. Since ρ is linear, and concatenated onto the top edge of $\mathscr{D}$, one can easily identify it. This construction is into $\mathscr{S}_{n+9}(h-1)$ and is at most n to 1, since one can identify ρ and attempt to put it back with σ' in at most n ways. Therefore

$$s_n(h, 0) \leq n s_{n+1}(h-1, 0) + n s_{n+9}(h-1, 0)$$

and by proposition 2.2(i), the inequality follows.

Remarks. By proposition 2.4(i) and (ii), in two dimensions, $s_n s_m \leq s_{n+m}$. Together with propositon 2.6 this implies that $\sup_{n>0} s_n^{1/n} = \lim_{n\to\infty} s_n^{1/n} = \beta$, where β is a finite, positive constant (Hille 1948). Similarly, $s_n(1, 0) s_m(1, 0) \leq s_{n+m}(1, 0)$ so that $\sup_{n>0} s_n(1, 0)^{1/n} = \lim_{n\to\infty} s_n(1, 0)^{1/n} = \beta_{1,0}$, where $\beta_{1,0}$ is a finite, positive constant. In addition, by propositions 2.2(ii), 2.6 and 3.3 it is obvious that $s_{n-hC}(1, 0) \leq s_n(h+1, 0) \leq [\Pi_{i=1}^{h} 2(n+(i-1)c_3)] s_{n+hc_3}(1, 0)$. If one takes the $(1/n)$th power, and let $n \to \infty$, then $\lim_{n\to\infty} s_n(h, 0)^{1/n} = \beta_{1,0}$. Also, it is easily calculated that $s_n(h, 0) \leq [2\beta_{1,0}^{c_3}(n+hc_3)]^{h-1} \beta_{1,0}^{n}$. A lower bound on $s_n(h, 0)$ can be calculated from propositions 2.5 and 3.1, using the theory on subadditive functions by Hammersley (1962). (If proposition 3.2 instead of 2.5 is used, then a stronger inequality results). Lastly, a consequence of proposition 3.2 is that $\beta_{1,0} < \beta$ (JRW1). I take these results together in theorem 3.4:

Theorem 3.4. Let $d = 2$. Then there exists finite, positive constants β, $\beta_{1,0}$ and c_i, $0 \leq i \leq 6$ such that

(i) $\sup_{n>0} s_n^{1/n} = \lim_{n\to\infty} s_n^{1/n} = \beta$,

(ii) $\lim_{n\to\infty} s_n(h,0)^{1/n} = \beta_{1,0}\ \forall h \geqslant 1$ and $s_n(h+1,0) \leqslant [2\beta_{1,0}^{c_3}(n+(h+1)c_3)]^h \beta_{1,0}^n$,

(iii) $s_{n+hc_2}(h+1,0) \geqslant c_4(c_1^{-h}/h!)(\lfloor n/c_0 \rfloor - h)^h n^{-c_5} \exp(-c^6(\log n)^2)\beta_{1,0}^n$, provided that $\lfloor n/c_0 \rfloor > h$, and

(iv) $\beta_{1,0} < \beta$.

Proof. (i), (ii) and (iv) are obvious or already proven. To see (iii), consider proposition 3.1(i), which is of the form

$$s_{n+m}(1,0) \leqslant g(n+m)s_n(1,0)s_m(1,0)$$

and $g(n+m)$ is given by

$$(\alpha_0(n+m))^{3\lceil \log(n+m)/\log(3/2) \rceil}.$$

Then,

$$(\log s_n(1,0))/n \geqslant \log \beta_{1,0} + (\log g(n))/n - 4 \sum_{m=2n}^{\infty} (\log g(m))/(m(m+1))$$

(Hammersley 1962), for all $n \geqslant 1$. To find an upper bound on the infinite sum, approximate it by an integral: note that the summand is a decreasing function with m, so that

$$\sum_{m=2n}^{\infty} (\log g(m))/(m(m+1)) \leqslant \int_{2n}^{\infty} (\log g(x-1))/(x(x-1))\,\mathrm{d}x$$

$$= \int_{2n-1}^{\infty} (\log g(x))/(x(x+1))\,\mathrm{d}x.$$

But

$$(\log g(x))/(x(x+1)) \leqslant (\log g(x))/x^2$$

and

$$\log g(x) \leqslant 3(\log(3x/2))^2/\log(3/2)$$

so that an upper bound on the sum would be

$$(9/(2\log(3/2))) \int_{3n-3/2}^{\infty} (\log x)^2/x^2\,\mathrm{d}x.$$

The solution to this integral can be found in Gradshteyn and Ryznik (1965). On simplification, one finds that

$$s_n(1,0) \geqslant c_4 n^{-c_5} \mathrm{e}^{-c_6(\log n)^2} \beta_{1,0}^n$$

where c_4, c_5 and c_6 are *positive* constants (as can be checked), and where I noted that $2n \geqslant 2n-1 \geqslant n$, and $3n - \frac{3}{2} \leqslant 3n$. Observe now that

$$s_{n+hc_2}(h+1,0) \geqslant (c_1^{-h}/h!)(\lfloor n/c_0 \rfloor - h)^h s_n(1,0)$$

when $\lfloor n/c_0 \rfloor > h$ (proposition 3.2).

Numerical simulations (Glaus 1986, 1988, Glaus and Einstein 1987) of surfaces suggest the existence of a *critical exponent* ϕ such that $s_n \sim c_7 n^{-\phi}\beta^n$ (where $a_n \sim f(n)$ means that $a_n = f(n)[1+\mathrm{o}(1)]$). Assume that $s_n(h,0) \sim c_7(h,0)n^{-\phi_{h,0}}\beta_{1,0}^n$. Then one can relate $\phi_{h,0}$ to $\phi_{1,0}$, and find a bound on $\phi_{1,0}$ by substituting the assumption into the inequalities derived in the propositions above.

Theorem 3.5. Let $d=2$. Suppose that $s_n(h,0) \sim c_7(h,0)n^{-\phi_{h,0}}\beta_{1,0}^n$. Then

(i) $\phi_{h+1,0} = \phi_{1,0} - h$,

(ii) $\phi_{1,0} \geq 0$, and

(iii) $(2\beta_{1,0}^{c_3})^{-h} \leq c_7(1,0)/c_7(h+1,0) \leq h!(c_1\beta_{1,0}^{c_2})^h$.

Proof. (i) By propositions 3.2 and 3.3

$$c_1^{-h}\binom{\lfloor (n-c_2h)/c_0 \rfloor}{h} s_{n-c_2h}(1,0) \leq s_n(h+1,0) \leq [2(n+hc_3)]^h s_{n+hc_3}(1,0).$$

Substitute the assumption, divide by $\beta_{1,0}^n$, take logarithms, divide by $\log n$ and let $n \to \infty$. If one keeps in mind that $\binom{an}{b} \sim (an)^b$, then one finds that $\phi_{h+1} = \phi_1 - h$.

(ii) Observe that $s_n(1,0) \leq \beta_{1,0}^n$, this implies that $\phi_1 \geq 0$.

(iii) Substitute the assumption into proposition 3.2 and 3.3, with $\phi_{h+1} = \phi_1 - h$, and observe that $\binom{m}{h} \geq (m-h)^h/h!$. Divide the resulting equation by $n^{h-\phi_1}$ and let $n \to \infty$. This gives the desired results.

The results $\phi_{h+1} = \phi_1 - h$ was also proven for lattice animals, where cycles are counted instead of boundary components (Soteros and Whittington 1988).

3.2. Surgery in more than two dimensions

In three and higher dimensions there is the possibility that a surface will have genus more than zero, in addition to boundary components which may be present. I begin by generalizing two constructions in JRW1 and JRW2.

Proposition 3.6. Let $d \geq 3$, and suppose that $h \geq 1$. Then

$$s_n(h+1,g) \leq n \sum_{j=1}^{\lceil n/4 \rceil} \sum_{g'=0}^{g} s_{n-j}(h, g-g') s_j(1,g').$$

Proof. Consider the shortest connected sequence of plaquettes which originates on one boundary component, and terminates on another, in the surface $\sigma_n \in \mathcal{S}_n(h,g)$, where $h \geq 2$. By the arguments in proposition 2.3 this sequence has (say) j plaquettes, where $1 \leq j \leq \lceil n/4 \rceil$. The maximum number of possible configurations of these j plaquettes, which have one boundary component (by the construction), is $s_j(1,g')$ (where $0 \leq g' \leq g$). Therefore, the construction is a map

$$\mathcal{S}_n(h,g) \to \bigcup_{j<\lceil n/4 \rceil} \bigcup_{g' \leq g} \mathcal{S}_{n-j}(h-1, g-g') \times \mathcal{S}_j(1,g')$$

which is at most n to 1. (To see this, consider a strip of j plaquettes which is being moved around the boundary components of surfaces in $\mathcal{S}_{n-j}(h-1)$. If it separates a boundary component into two components, then a successful fit is obtained. If j plaquettes are involved in a successful fit, then every plaquette in the sequence, except perhaps for the first and the last, must have two edges in the boundary. Therefore, the maximum number of ways that one can put back the connected strip is bounded above by $2(n-j+1)/2$. This is a maximum if $j=1$.) Hence, $s_n(h,g) \leq n\sum_{j=1}^{\lceil n/4 \rceil} \sum_{g'=0}^{g} s_j(1,g') s_{n-j}(h-1, g-g')$.

Proposition 3.7. Let $d \geq 3$, and let $g \geq 0$. Then

$$s_n(1,g+1) \leq n \sum_{j=1}^{\lceil n/2 \rceil} \sum_{g'=0}^{g} s_{n-j}(2,g') s_j(1,0).$$

Proof. Let $\sigma_n \in \mathcal{S}_n(1, g)$ where $g \geqslant 1$. Then σ_n is homeomorphic to an orientable 2-manifold with g handles and one boundary component. σ_n can also be considered a cellular complex consisting of plaquettes, edges and vertices. Consider the set $\mathscr{C}$ of all 1-cycles which are not null-homologous and which has at least one edge in $\partial\sigma_n$. Define the length of a cycle in $\mathscr{C}$ to be the number of edges it has not in $\partial\sigma_n$. Let $C \in \mathscr{C}$ be a cycle with minimum length. On either side of C there are strips of plaquettes, and which start and terminate in the boundary of σ_n. If one deletes one of these, then C becomes a boundary (null-homologous), and the genus of σ_n will decrease by at least one. Let the strip contain j plaquettes. Obviously, $j < \lceil n/2 \rceil$, since one can always elect to delete the plaquettes in the shorter strip along C. The strip which is removed has zero genus (to see this, observe that if the strip has a handle, then one can cut that instead, removing less plaquettes than j). Moreover, there are now two boundary components on the rest of σ_n (this follows from an easy application of the Jordan-Brouwer curve theorem (see Greenberg and Harper 1981) to the neighbourhood of C), while the strip has only one boundary component. The construction is a map $\mathcal{S}_n(1, g) \to \bigcup_{j \leqslant \lceil n/2 \rceil} \bigcup_{g' < (g-1)} \mathcal{S}_{n-j}(2, g') \times \mathcal{S}_j(1, 0)$, which is at most n to 1 (see proposition 3.6). Hence, $s_n(1, g) \leqslant n \Sigma_{j=1}^{\lceil n/2 \rceil} \Sigma_{g'=0}^{g-1} s_{n-j}(2, g') s_j(1, 0)$, which is the desired result.

Remarks. The existence of growth constants for open surfaces can be proven in an elegant fashion. Consider for example proposition 2.3(i) and proposition 2.4(i): together, they give $s_n s_m \leqslant 4n^{\lceil \log n/\log 4 \rceil} m^{\lceil \log m/\log 4 \rceil} s_n^t s_m^b \leqslant 4n^{\lceil \log n/\log 4 \rceil} m^{\lceil \log m/\log 4 \rceil} s_{n+m}$. Consequently, if proposition 2.6 is kept in mind, then there exists a positive constant β such that $\lim_{n\to\infty} s_n^{1/n} = \beta$ exists (Hammersley 1962). While these inequalities suffice to prove the existence of the growth constant, the strongest bound on the approach to the limit is found by a concatenation developed in JRW1:

Proposition 3.8. (JRW1) Let $d \geqslant 3$. Then there exists a constant k_0 such that

(i) $s_n s_m \leqslant s_{n+m+k_0}$, and

(ii) $s_n(h_1, g_1) s_m(h_2, g_2) \leqslant s_{n+m+k_0}(h_1 + h_2, g_1 + g_2)$.

Remarks. A simple extension of the theory of subadditive functions, due to Wilker and Whittington (1979), when considered with proposition 3.8(i), implies then that $\lim_{n\to\infty} s_n^{1/n} = \beta$ exists and $s_n \leqslant \beta^{n+k_0}$. Similar deductions can be made about *discs* (elements of $\mathcal{S}_n(1, 0)$). The existence of the limit $\lim_{n\to\infty} s_n(1, 0)^{1/n} = \beta_{1,0}$ follows from proposition 3.1(ii) and proposition 2.6 (Hammersley 1962). Here $\beta_{1,0}$, is the growth constants for discs, which is presumable different from β (as numerical work seems to indicate (Glaus 1986, 1988, Glaus and Einstein 1987)), but a rigorous proof of this fact has not yet been found. A lower bound on the rate of convergence to the limit can be derived from proposition 3.1, as was done in two dimensions in proposition 3.4. An upper bound is found by noting that $s_n(1, 0) s_m(1, 0) \leqslant 4(d-1) n^{\lceil \log n/\log 4 \rceil} m^{\lceil \log m/\log 4 \rceil} s_{n+m}(1, 0)$ (which follows from propositions 2.3(iii) and 2.4(ii)). Unfortunately, the methods here do no provide a lower bound on the rate of convergence of $s_n^{1/n}$ to β; to find such a bound an inequality like that for discs in proposition 3.1 must be proven. These results are taken together in theorem 3.9:

Theorem 3.9. Let $d \geqslant 3$. Then there exists positive constants $\beta, \beta_{1,0}$ and k_i, $0 \leqslant i \leqslant 7$, such that

(i) $\lim_{n\to\infty} s_n^{1/n} = \beta$ and $s_n \leqslant \beta^{n+k_0}$,

(ii) $\lim_{n\to\infty} s_n(1, 0)^{1/n} = \beta_{1,0}$,

(iii) $s_n(1, 0) \leqslant k_1 n^{k_2} \exp(k_3 (\log n)^2) \beta_{1,0}^n$, and $s_n(1, 0) \geqslant k_4 n^{-k_5} \exp(-k_6 (\log n)^2) \times \exp(-k_7 (\log n)^3) \beta_{1,0}^n$.

Proof. (i) and (ii) were proven in the above discussion. All that are left are the bounds in (iii).

First the upper bound: by propositions 2.3(iii) and 2.4(ii) one has

$$s_n(1,0)s_m(1,0) \leq 4(d-1)n^{\lceil \log n/\log 4\rceil} m^{\lceil \log m/\log 4\rceil} s_{n+m}(1,0).$$

In this inequality, let

$$q_n = s_n(1,0)/(2\sqrt{d-1}\; n^{\lceil \log n/\log 4\rceil}).$$

Then

$$\log q_{n+m} \geq \log q_n + \log q_m + \log g(n+m)$$

where

$$\log g(n) = -\log(2\sqrt{d-1}) - \log n \lceil \log n/\log 4\rceil.$$

Then,

$$\log q_n/n \leq \beta_{1,0} + \log g(n)/n - 4\sum_{m=2n}^{\infty} \log g(m)/(m(m+1))$$

(Hammersley 1962). The theorem is proven if one finds an upper bound on $\log g(n)/n$ and a lower bound on the sum. Since $\log g(n)$ is negative, an upper bound on

$$\sum_{m=2n}^{\infty} (\log(2\sqrt{d-1}) + \log m \lceil \log m/\log 4\rceil)/(m(m+1))$$

is desired. This sum is bounded by the integral

$$\int_{2n-1}^{\infty} (\log(2\sqrt{d-1})/x^2 + \log x/x^2 + (\log x/x)^2/\log 4)\; \mathrm{d}x.$$

Observe that the integral is monotonic decreasing with increasing n. An upper bound (which is sufficient) is then found by replacing $2n-1$ with n, and evaluating the integral (Gradshteyn and Rysnik 1965) to find

$$((\log(2\sqrt{d-1}) + 2/\log 4 + 1) + (2/\log 4 + 1)\log n + (\log n)^2/\log 4)/n.$$

The upper bound on $\log g(n)/n$ is found by noting that

$$\log n \lceil \log n/\log 4\rceil \geq (\log n)^2/\log 4.$$

Collecting terms gives

$$\log s_n(1,0) \leq n \log \beta_{1,0} + (2(\log n)^2)/\log 4 + (4 + 8/\log 4)\log n$$
$$+ (2\log(2\sqrt{d-1}) + 1 + 2/\log 4).$$

Put

$$\log k_1 = 2\log(2\sqrt{d-1}) + 1 + 2/\log 4,\; k_2 = 4 + 8/\log 4$$

and $k_3 = 2/\log 4$. This is the desired upper bound.

To derive the lower bound we follow a similar argument to that in theorem 3.4(iii): from proposition 3.1(ii) observe that

$$s_{n+m}(1,0) \leq g(n+m)s_n(1,0)s_m(1,0)$$

and a find a lower bound by noting that

$$\log s_n(1,0)/n \geq \log \beta_{1,0} + \log g(n)/n - 4\sum_{m=2n}^{\infty} \log g(m)/(m(m+1))$$

where

$$g(n)=(\alpha_1(d-1)n^{3+2\lceil \log n/\log 4\rceil})^{\lceil \log n/\log(3/2)\rceil}.$$

We seek a lower bound on $\log g(n)/n$, and an upper bound on $\Sigma_{m=2n}^{\infty} \log g(m)/(m(m+1))$. After a significant amount of algebra, where I note that $2n \geqslant 2n-1 \geqslant n$, and where I bound the infinite sum by an integral as in theorem 3.4(iii), one gets

$$\log s_n(1,0) \geqslant n \log \beta_{1,0} - \log k_4 - k_5 \log n - k_6(\log n)^2 - k_7(\log n)^3.$$

The numbers k_i, where $4 \leqslant i \leqslant 7$, are positive constants (this is easy but tedious to check).

Remarks. Theorem 3.9(iii) gives bounds on the rate of approach of the limit in theorem 3.9(ii). These results are only valid for embedded surfaces homeomorphic to a disc (or a punctured sphere). The effects of more boundary components or handles can now be considered if we use propositions 3.6 and 3.7 and the unfolding of surfaces.

Theorem 3.10. Let $d \geqslant 3$. Then
(i) $\lim_{n\to\infty} s_n(h, g)^{1/n} = \beta_{1,0}$ exists for any h and g, and
(ii) $s_n(h, g) \leqslant k_1(h, g) n^{k_2(h,g)} \exp(k_3(h, g)(\log n)^2)\beta_{1,0}^n$

and

$$s_{n+(h-1)H+gG}(h, g) \geqslant k_4 n^{-k_5} \exp(-k_6(\log n)^2 - k_7(\log n)^3)\beta_{1,0}^n$$

where the $k_i(h, g)$, $1 \leqslant i \leqslant 3$ are positive constants dependent only on h and g, and where the constants k_i, $4 \leqslant i \leqslant 7$ are the constants defined in theorem 3.9(iii).

Proof. (i) The proof is by induction. First I show that $\lim_{n\to\infty} s_n(1, g)^{1/n} = \beta_{1,0}$. By propositions 2.3(iii), 2.4(ii) and 3.7 we note that

$$s_n(1, g+1) \leqslant 4(d-1)n^{2+2\lceil \log n/\log 4\rceil} \sum_{g'=0}^{g} s_{n+C}(2, g').$$

Similarly, by propositions 2.3(iii), 2.4(ii) and 3.6 we find

$$s_n(h+1, g) \leqslant 4(h+1)(d-1)n^{2+2\lceil \log n/\log 4\rceil} s_{n+(h-1)C}(h, g).$$

But

$$s_n(1,0) \leqslant s_{n+hH+gG}(h+1, g).$$

Perform an induction now first on g, and then on h. Now for (ii): suppose that

$$s_n(1, g) \leqslant k_1(h, g) n^{k_2(h,g)} \exp(k_3(h, g)(\log n)^2)\beta_{1,0}^n$$

where $k_i(h, g)$ are constants depending only on h and g, for $1 \leqslant i \leqslant 3$, and for all h and for all $g \leqslant g_c$. Then observe that

$$s_n(1, g_c+1) \leqslant n \sum_{j=1}^{\lceil n/2\rceil} \sum_{g=0}^{g_c} s_{n-j}(2, g) s_j(1, 0)$$

(by proposition 3.7). Consequently,

$$s_n(1, g_c+1) \leqslant n\beta_{1,0}^n \sum_{j=1}^{\lceil n/2\rceil} \sum_{g=0}^{g_c} k_1(2, g) k_1(1, 0)(n-j)^{k_2(2,g)} j^{k_2(1,0)}$$
$$\times \exp(k_3(2, g)(\log(n-j))^2) + k_3(1, 0)(\log j)^2.$$

I can bound the last expression by noting that $n-j\leqslant n$ and $j\leqslant n$. Pick $k_1'(1,g_c+1)=\max_{0\leqslant g\leqslant g_c}\{k_1(1,0)\times k_1(2,g)\}$ and $k_i'(1,g_c+1)=\max_{0\leqslant g\leqslant g_c}\{k_i(1,0)+k_i(2,g)\}$ where $2\leqslant i\leqslant 3$. Then,

$$s_n(1,g_c+1)\leqslant k_1'(1,g_c+1)(g_c+1)n^{2+k_2'(1,g_c+1)}\exp(k_3'(1,g_c+1)(\log n)^2)\beta_{1,0}^n.$$

Thus one can perform an induction on $s_n(1,g)$ (with respect to g). Similarly, suppose that

$$s_n(h,g)\leqslant k_1(h,g)n^{k_2(h,g)}\exp(k_3(h,g)(\log n)^2)\beta_{h,0}^n$$

for all g and for all $h\leqslant h_c$. Then

$$s_n(h_c+1,g)\leqslant n\sum_{j=1}^{\lceil n/4\rceil}\sum_{g'=0}^{g}s_{n-j}(h_c,g-g')s_j(1,g')$$

(by proposition 3.6). Thus

$$s_n(h_c+1,g)\leqslant n\beta_{1,0}^n\sum_{j=1}^{\lceil n/4\rceil}\sum_{g'=0}^{g}k_1(h_c,g-g')k_1(1,g')(n-j)^{k_2(h_c,g-g')}j^{k_2(1,g')}$$
$$\times\exp(k_3(h_c,g-g')(\log(n-j))^2+k_3(1,g')(\log j)^2).$$

Observe that $n-j\leqslant n$ and $j\leqslant n$, and define

$$k_1'(h_c+1,g)=\max_{0\leqslant g'\leqslant g}\{k_1(h_c,g')\times k_1(1,g')\},$$

and

$$k_i'(h_c+1,g)=\max_{0\leqslant g'\leqslant g}\{k_i(h_c,g')+k_i(1,g')\}.$$

Then

$$s_n(h_c+1,g)\leqslant(g+1)k_1'(h_c+1,g)n^{2+k_2'(h_c+1,g)}\exp(k_3'(h_c+1,g)(\log n)^2\beta_{1,0}^n.$$

The upper bound follows then by induction. The lower bound is found by noting that $s_n(1,0)\leqslant s_{n+hH+gG}(h+1,g)$ (proposition 2.5(ii) and (iii)).

Remarks. Consider the number of surfaces with h boundary components, $s_n(h,*)$-($=s_n(h)$ in JRW1. There is ample numerical evidence that $s_n(h,*)\sim k_8(h,*)n^{-\phi_h}\beta_1$ (the existence of β_1 is easily shown using the same methods as in theorem 3.9) where it is strongly believed that $\phi_1\geqslant 1$ in every dimension (Bovier *et al* 1984, Fröhlich 1985, Glaus 1986, Glaus and Einstein 1987). Proposition 2.5(ii) implies that $\phi_h\geqslant\phi_{h+1}$. If one sum both sides of proposition 3.6 over g, then $s_n(h+1,*)\leqslant n\sum_{j=0}^{\lceil n/4\rceil}s_{n-j}(h,*)s_j(1,*)$. Substitute the assumption into this expression and observe that $\phi_1\geqslant 1$. Then we recover (1.4). Several other relations among the exponents were derived in JRW1 and JRW2. It is instructive to derive equation (4.2) in JRW2 from the generalized equations in this paper. I do that by proving a series of lemmas involving inequalities between the exponents, where I assume that $s_n(h,g)\sim k_8(h,g)n^{-\phi_{h,g}}\beta_{10}$, and $\phi_{1,0}\geqslant 1$ in every dimension.

Lemma 3.11. If $d\geqslant 3$, then $\phi_{h,0}\geqslant\phi_{h-1,0}-2$.

Proof. Substitute the assumptions into proposition 3.6, with $g=0$. This gives

$$s_n(h+1,0)\leqslant k_8(h+1,0)n\sum_{j=1}^{\lceil n/4\rceil}(n-j)^{-\phi_{h,0}}j^{-\phi_{1,0}}\beta_{1,0}^n(1+o(1)).$$

Bound $j^{-\phi_{1,0}}$ by 1. The bound the sum to find

$$s_n(h+1,0) \leq k_8(h+1,0)n\lceil n/4\rceil(1+o(1))O(n^{-\phi_{h,0}})\beta_{1,0}^n.$$

Divide this inequality by $\beta_{1,0}^n$, take logarithms, divide by $\log n$ and let $n \to \infty$.

Lemma 3.12. If $d \geq 3$, then $\phi_{1,g+1} \geq \phi_{2,g} - 2$.

Proof. Apply proposition 2.5(i) and (iii) to proposition 3.7 so that

$$s_n(1, g+1) \leq n(g+1) \sum_{j=1}^{\lceil n/2\rceil} s_{n-j+gGA}(2, g)s_j(1, 0).$$

Substitute the assumptions and argue as in lemma 3.11.

Lemma 3.13. If $d \geq 3$, then $\phi_{2,g} \geq \phi_{1,0} - 2 - 4g$.

Proof. The proof is by induction. Firstly, let $g = 1$. Then observe that

$$s_n(2,1) \leq n \sum_{j=1}^{\lceil n/4\rceil} \{s_{n-j}(1,1)s_j(1,0) + s_{n-j}(1,0)s_j(1,1)\}$$

(proposition 3.7). Consequently, if one substitutes the assumption,

$$s_n(2,1) \leq k_8(1,1)k_8(1,0)n \sum_{j=1}^{\lceil n/4\rceil} \{(n-j)^{-\phi_{1,1}}j^{-\phi_{1,0}} + (n-j)^{-\phi_{1,0}}j^{-\phi_{1,1}}\}\beta_{1,0}^n(1+o(1)).$$

But by lemmas 3.11 and 3.12 $\phi_{1,1} \geq \phi_{2,0} - 2 \geq \phi_{1,0} - 4$. Substitute this into the inequality and bound $j^{-\phi_{1,0}}$ by 1 in both terms. The sequence of operations in lemmas 3.11 (or in lemma 3.12) then gives $\phi_{2,1} \geq \phi_{1,0} - 6$. Secondly, suppose now that $\phi_{2,g-1} \geq \phi_{1,0} - 2 - 4(g-1)$. Then $\phi_{1,g} \geq \phi_{2,g-1} - 2 \geq \phi_{1,0} - 4g$ (by lemmas 3.11 and 3.12). Then, by the assumptions and by proposition 3.7;

$$s_n(2,g) \leq n \sum_{j=1}^{\lceil n/2\rceil} \sum_{g'=0}^{g} \{k_8(1, g-g')k_8(1,g')(n-j)^{4(g-g')-\phi_{1,0}}j^{4g'-\phi_{1,0}}\}\beta_{1,0}^n(1+o(1)).$$

Observe that

$$(n-j)^{4(g-g')-\phi_{1,0}}j^{4g'-\phi_{1,0}} \leq n^{4g-\phi_{1,0}}$$

so that

$$s_n(2,g) \leq n\lceil n/2\rceil \sum_{g'=0}^{g} \{k_8(1,g-g')k_8(1,g')\}n^{4g-\phi_{1,0}}\beta_{1,0}^n(1+o(1)).$$

Hence, $\phi_{2,g} \geq \phi_{1,0} - 2 - 4g$, and the induction on g proves the lemma.

The inequality can now be proven:

Theorem 3.14. If $d \geq 3$, then $\phi_{h,g} \geq \phi_{h+1,g}$, $\phi_{h,g-1} \geq \phi_{h,g}$, and $\phi_{h+1,g} \geq \phi_{1,0} - 2h - 4g$.

Proof. The first two inequalities follows from concatenation (proposition 2.5). The last inequality is proven by induction on h. By lemma 3.13, $\phi_{2,g} \geq \phi_{1,0} - 2 - 4g$. Suppose that $\phi_{h,g} \geq \phi_{1,0} - 2(h-1) - 4g$. By the hypothesis and the assumptions, I find

$$s_{n+1}(h+1,g) \leq n \sum_{j=1}^{\lceil n/4\rceil} \sum_{g'=0}^{g} \{k_8(h, g-g')k_8(1,g')(n-j)^{2(h-1)+4(g-g')-\phi_{1,0}}j^{4g'-\phi_{1,0}}\}$$
$$\times \beta_{1,0}^n(1+o(1)).$$

Observe that

$$(n-j)^{2(h-1)+4(g-g')-\phi_{1,0}}j^{4g'-\phi_{1,0}} \leq n^{2(h-1)+4g-\phi_{1,0}}$$

whence

$$s_n(h+1,g) \leq n\lceil n/4 \rceil \sum_{g'=0}^{g} \{k_8(h,g-g')k_8(1,g')\} n^{2(h-1)+4g-\phi_{1,0}} \beta_{1,0}^n(1+o(1)).$$

Divide this inequality by $\beta_{1,0}^n$, take logarithms and divide by $\log n$. Then take $n \to \infty$.

4. Conclusions

There is much which is not known about surfaces, and many speculations, derived mostly from non-rigorous analysis, such as the renormalization group or numerical simulations, remain unproven. The constructive methods in this paper revealed some interesting facts about $s_n(h, g)$ and were slightly more successful in two dimensions (than in higher dimensions). Some of the puzzles which remains unproven are:

(i) Can one prove that $c_3 = 0$ in theorem 3.4, and that $k_3 = k_6 = k_7 = 0$ in theorem 3.9? I believe that this is easier in the two-dimensional case. In that case, if $c_3 = 0$, one notes that

$$\begin{aligned} h - c_5 &\leq \liminf_{n\to\infty} (\log(s_n(h+1,0)/\beta_{1,0}^n)/\log n) \\ &\leq \limsup_{n\to\infty} (\log(s_n(h+1,0)/\beta_{1,0}^n)/\log n) \leq h. \end{aligned}$$

Thus, the correction to the exponential growth of $s_n(h, 0)$ is dominated at most by a power law, and $\phi_{h,0}$ can be defined as the limsup in the above expression. The same arguments can be made in three and higher dimensions about $\phi_{h,g}$.

(ii) It is known rigorously that $\beta > \beta_{1,0}$ in two dimensions (theorem 3.4(iv) and JRW1). Can this result be proven in three and higher dimensions? It seems very plausible.

(iii) Theorem 3.5(ii) states that $\phi_{1,0} \geq 0$ in two dimensions (if it exists). Can this be proven in three and higher dimensions? I noted in section 1 that there is ample evidence that this is indeed the case in every dimension. Also, can the exponents in three and higher dimensions be related to each other as in theorem 3.5(*i*)? The conjecture (that $\phi_{h+1} = \phi_1 - h$ in three and higher dimensions) in JRW1 remains unproven.

(iv) Closed surfaces (i.e. those in the sets $\mathscr{S}_n(0, g)$, where $g \geq 0$) have resisted all attempts at yielding results. Obviously, $\lim_{n\to\infty} s_n(0,g)^{1/n} = \beta_{0,0}$ exists if $g = 0$, but what if $g \neq 0$? I can show that $\beta_{0,0} < \beta$; this is fairly easy to prove: we must prove a relationship like proposition 3.2 to relate $s_n(0, 0)$ and $s_n(h, 0)$, the result follows immediately. Can one prove that $\beta_{0,0} < \beta_{0,*}$ (or $\beta_{0,0} < \beta_{1,0}$, or $\beta_{1,0} < \beta$)? Paradoxically, one can obtain some interesting results if closed surfaces are counted by volume (instead of by surface area) (Janse van Rensburg 1990).

(v) In JRW2 we proved relations among several exponents, and conjectured many more, involving the knots in the boundary components. Can one prove any of these?. The remarks made in JRW2 apply.

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